

Competition in the presence of aging: order, disorder, and synchronized collective behavior

Toni Pérez,¹ Konstantin Klemm,^{2,3,4,5} and Víctor M. Eguíluz¹

¹*Instituto de Física Interdisciplinar y Sistemas Complejos (IFISC), Spain*

²*School of Science and Technology, Nazarbayev University,
Kabanbay Batyr Ave. 53, 010000 Astana, Kazakhstan*

³*Bioinformatics, Institute of Computer Science, University Leipzig, Härtelstr. 16-18, 04107 Leipzig, Germany*

⁴*Bioinformatics and Computational Biology, University of Vienna, Währingerstraße 29, 1090 Vienna, Austria*

⁵*Theoretical Chemistry, University of Vienna, Währingerstraße 17, 1090 Vienna, Austria*

We study the stochastic dynamics of coupled states with transition probabilities depending on local persistence, this is, the time since a state has changed. When the population has a preference to adopt older states the system orders quickly due to the dominance of the old state. When preference for new states prevails, the system can show coexistence of states or synchronized collective behavior resulting in long ordering times. In this case, the magnetization $m(t)$ of the system oscillates around $m(t) = 0$. Implications for social systems are discussed.

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Models of two states are commonly used in physics as a tool to study the emergence of collective behavior in systems from spin interaction to opinion dynamics [1–3]. In the adoption of traits [4–7] different aspects have been studied including the relevance of the interaction topology [8–10], social influence [11, 12], and mass media [13–15]. When accounting for opinion dynamics, the majority of models are based on decision rules that consider a fraction of the surrounding states, e.g., voter model [16], threshold model [17], majority rule [18], or Sznajd model [19]. The timing of the interactions can also affect the behavior of the system at least by two ways: the precise sequence of interactions and by the aging of states. For example, in epidemic spreading and diffusion, the temporal sequence of interactions can slow down the spreading process [20–23]; in ordering dynamics, state-dependent updates can have a qualitative impact on the mean time to order [24–29]. Aging in physical systems refers to the persistence time, that is, the time spent in a given state, and affects the response of the system to an external field or perturbation [30, 31]. In social systems, when individuals make choices they usually rely on their own past experience or memory [32–34]. While conservative groups tend to hold ideas in an unaltered form for a long time, progressive individuals embrace new opinions, ideas, or a technology and disseminate them with more enthusiasm [35, 36]. In the competition between new and old information, although new information is more valuable for exploring and spatial searching [37], adopting older strategies can promote cooperation and group success [38]. Also in a biological context, aging in speciation events has been proposed as a mechanism to explain the shape of evolutionary trees [39].

Here we analyze how the tendency of agents towards the adoption of established vs. novel traits influences the macroscopic dynamic and the ordering process. We tackle this problem by considering a model in which the

adoption of states depends on the time span the agents have held their current states.

The model is defined as follows: each agent has a state l that can be up (\uparrow) or down (\downarrow) with age young (y) or old (z). Agents can be then in four states y^\uparrow , y^\downarrow , z^\uparrow , and z^\downarrow . Young agents become old at a rate that we set to $r = 1$. Then, there are reactions of randomly paired agents: i) in young i and old j of opposite opinions, i adopts the opinion of j with probability $w = \frac{1}{2} + \epsilon$, otherwise (with probability $1 - w$), j adopts the opinion of i ; ii) in pairs of agents with the same age and different opinion, each agent has probability $\frac{1}{2}$ of convincing the other; iii) for pairs of agents with the same opinion, nothing happens. When an agent adopts an opinion, it goes to the young age of the adopted opinion. Neglecting correlations, the expectation values of state concentrations evolve according to

$$\begin{aligned}\dot{y}^\uparrow &= -y^\uparrow + (1 - 2w)y^\uparrow z^\downarrow + w y^\downarrow z^\uparrow + \frac{1}{2} z^\uparrow z^\downarrow, \\ \dot{y}^\downarrow &= -y^\downarrow + w y^\uparrow z^\downarrow + (1 - 2w)y^\downarrow z^\uparrow + \frac{1}{2} z^\uparrow z^\downarrow, \\ \dot{z}^\uparrow &= y^\uparrow - (1 - w)y^\downarrow z^\uparrow - \frac{1}{2} z^\uparrow z^\downarrow, \\ \dot{z}^\downarrow &= y^\downarrow - (1 - w)y^\uparrow z^\downarrow - \frac{1}{2} z^\uparrow z^\downarrow,\end{aligned}\tag{1}$$

with the normalization $y^\uparrow + y^\downarrow + z^\uparrow + z^\downarrow = 1$. Here we use $y^{\uparrow\downarrow}$ and $z^{\uparrow\downarrow}$ to refer to the fraction of the corresponding states occupied by the agents. The parameter ϵ corresponds to the persuasiveness of the agent, $\epsilon > 0$ means that agents with older opinions are more persuasive. On the contrary, $\epsilon < 0$ corresponds to agents with younger opinions been more persuasive.

The system presents three stationary solutions in the relevant range of all four variables being non-negative. Two fixed points are the homogeneous solutions S_1 having $z^\downarrow = 1$ and S_2 having $z^\uparrow = 1$. Here either all opinions are down (S_1) or all are up (S_2) and old. The homoge-

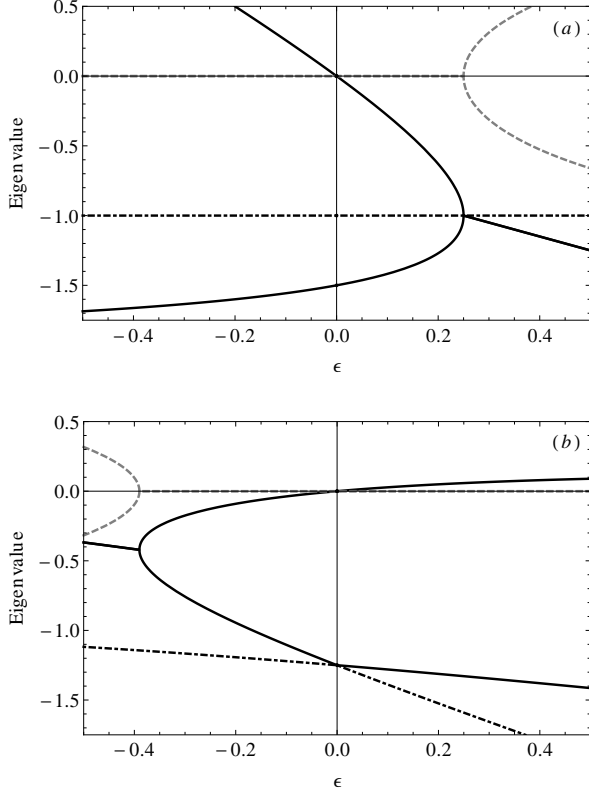


FIG. 1. Eigenvalues of the stationary solutions of Eq. (1) as a function of persuasiveness ϵ for: (a) the homogeneous solutions S_1 and S_2 having the same eigenvalues, and (b) the solution S_3 . Black solid (gray dashed) lines represent real (imaginary) parts of the two complex conjugate eigenvalues. Dotted-dashed black line represent the third eigenvalue (real). The fourth eigenvalue (not shown) for the eigenspace in $(1, 1, 1, 1)$ direction is zero due to conservation of normalization.

neous solutions are stable if $\epsilon > 0$. Non-zero imaginary parts of two eigenvalues are obtained for $\epsilon > 1/4$. The third fixed point S_3 is an up-down-symmetric solution with values $y^\uparrow = y^\downarrow = \frac{5+2\epsilon-\Delta}{8\epsilon}$, $z^\uparrow = z^\downarrow = \frac{-5+2\epsilon+\Delta}{8\epsilon}$ where $\Delta = \sqrt{25 + 4(\epsilon^2 + 3\epsilon)}$. As shown in Figure 1(b), it is stable if $\epsilon < 0$, thus complementary to the stability of the homogeneous solutions. A transition from zero to non-zero imaginary parts of two eigenvalues occurs when ϵ falls below approximately -0.39 . In this regime of strongly negative ϵ , the system oscillates when relaxing from a perturbation out of the symmetric fixed point solution S_3 . This stability scenario is qualitatively maintained when r changes. As $r \rightarrow 0$, the point at which the non-zero imaginary part of the eigenvalues shows up shifts towards $\epsilon = 0$.

Model with continuous ages. We now move from two ages to a continuous age space and introduce an age dependent probability. The model is now described as fol-

lows. Agents can be in one of the two opinions, up or down. The state of each agent has associated an age defined as: $\tau_i = t - t_i$, where t is the current time and t_i is the time when the current state of agent i was acquired. The system evolves by randomly selecting a pair of agents that, if they are in different states, with probability $p_{i \rightarrow j}$ agent j adopts the state of agent i , and with probability $1 - p_{i \rightarrow j}$ the contrary happens i copies the state of j . If the agents are already in the same state, no change is applied. After N updates, time t is increased to $t + 1$. We define the probability $p_{i \rightarrow j}$ to be dependent on the age of the states of both interacting agents as

$$p_{i \rightarrow j} = \frac{1}{1 + (\tau_j/\tau_i)^\alpha}. \quad (2)$$

Initially, each agent has randomly assigned one of the two opinions and the initial ages t_i are uniformly distributed proportionally to the system size N . We consider random mixing where each agent is allowed to interact with any other agent. The case $\alpha = 0$ corresponds to an updating probability of $p_{i \rightarrow j} = 0.5$ which leads to voter model dynamics [16]. Large values of the exponent ($\alpha \rightarrow \infty$), correspond to situations in which the agent with the initial oldest state is imposing her opinion to the others. In the other extreme case ($\alpha \rightarrow -\infty$), the youngest state is imposed.

For $\alpha \rightarrow \infty$ the system ends up in the state of the oldest opinion while for values of $\alpha \in (-\infty, 0]$ the system adopts any of the two opinions with equal probability. For $\alpha \in (0, \infty)$ the probability that the system adopts the state of the initial oldest opinion grows with increasing α but it tends to $1/2$ when N increases.

Figure 2 shows the probability distribution function of the density of states ρ as a function of α . For $\alpha = 0$, the density of states is homogeneous corresponding to an equiprobable distribution of states. For α negative but close to zero, the dynamic is concentrated around $\rho = 0.5$, which corresponds to a configuration where the agents alternate between any of the two opinions. This situation changes gradually to a more homogeneous distribution of states as α becomes more negative. For $\alpha > 0$, the states are concentrated around $\rho = 0$ and $\rho = 1$ showing that the system eventually orders in one of the two opinions (the presence of the two peaks is due to the alternation in being the oldest opinion during the initial conditions).

Figure 3 shows the ordering time $S_N(\alpha)$, i.e., the time that the system needs to reach a final state where all the agents have the same opinion, computed as the median of the distribution of ordering times from different simulations and rescaled to the value $S_N(\alpha = 0)$. $S_N(0)$ increases linearly with the N as it does for the voter model [9, 40]. For values $\alpha > 0$, $S_N(\alpha)$ gets smaller than $S_N(0)$ implying that the system orders faster than in the voter model case. There is a transition when α crosses zero. For values $\alpha \lesssim 0$, $S_N(\alpha)$ increases very fast with N .

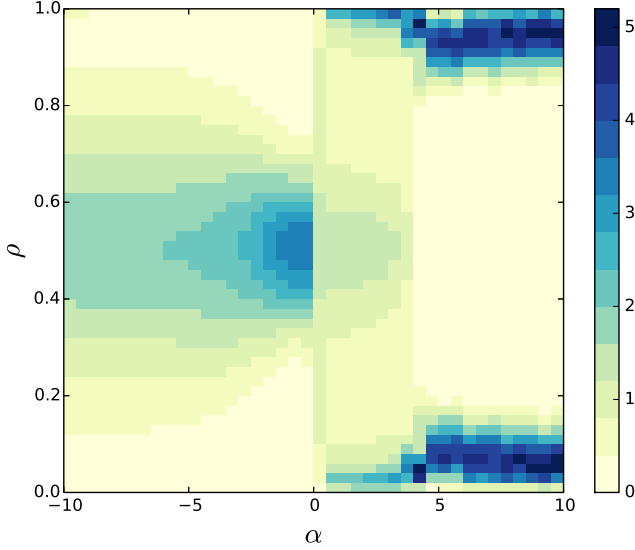


FIG. 2. Probability density function (codified as a colormap) of the dependence of the density of states with α . The density of states is calculated by computing the normalized cumulative histogram of the fraction of population in each state during a fixed simulation period and averaged over 10^4 realizations. The system size was fixed to $N = 100$.

This is in agreement with the observed dynamics around $\alpha = -1$ (see Fig. 2). The extreme values of α correspond to the cases where, when confronting two states, the oldest opinion always induces the change ($\alpha = +\infty$) or the youngest opinion always induces the change ($\alpha = -\infty$). The inset of Fig 3 shows the scaling with system size of $S_N(\alpha)$ in these limits. For $\alpha = +\infty$ the ordering time scales as $S_N \sim N^\gamma$ with the exponent $\gamma = 1.2$. In the other limit, for $\alpha = -\infty$, the ordering time scales as $S_N \sim N \exp(bN)$ with $b = 0.009$.

In the regime $\alpha < 0$, what is the behaviour of the system during the long ordering times? Figure 4(a) shows time series of the magnetization of the system. For α negative and sufficiently far from zero, the magnetization oscillates around zero. Figure 4(b) provides further analysis by the autocorrelation functions of the magnetization time series. The onset of oscillations is observed when α passes a value around -0.5 from above. Figure 4(c) shows frequency ω and decay constant γ extracted from the autocorrelation functions. These values do not exhibit significant dependence on system size. The decay constant is maximum at the onset of oscillations, i.e. where the frequency ω becomes non-zero. Both the onset of oscillations and the decay behaviour are captured by the basic model, cf. Figure 1(b). At the transition to non-zero imaginary parts (oscillations), the stability of the symmetric fixed point solution (S_3) is maximal, meaning that perturbations decay fastest.

To understand further the dynamic around $\alpha = 0$ we define a quantity called the convincingness z . Let

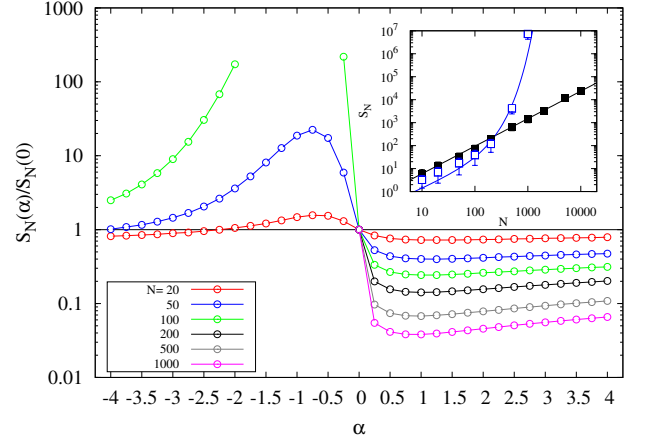


FIG. 3. Rescaled ordering time $S_N(\alpha)/S_N(0)$ versus α for different system sizes. Open symbols stand for the median of the ordering time normalized to the median of the ordering time at $\alpha = 0$. The horizontal black line has been added for visualization purposes. Inset: Scaling of the median of $S_N(\alpha)$ in the limits $\alpha \rightarrow \infty$ (solid symbols) and $\alpha \rightarrow -\infty$ (open symbols). Dashed lines fit respectively $S_N(+\infty) \sim N^\gamma$ with $\gamma = 1.2$ and $S_N(-\infty) \sim N \exp(bN)$ with $b = 0.009$.

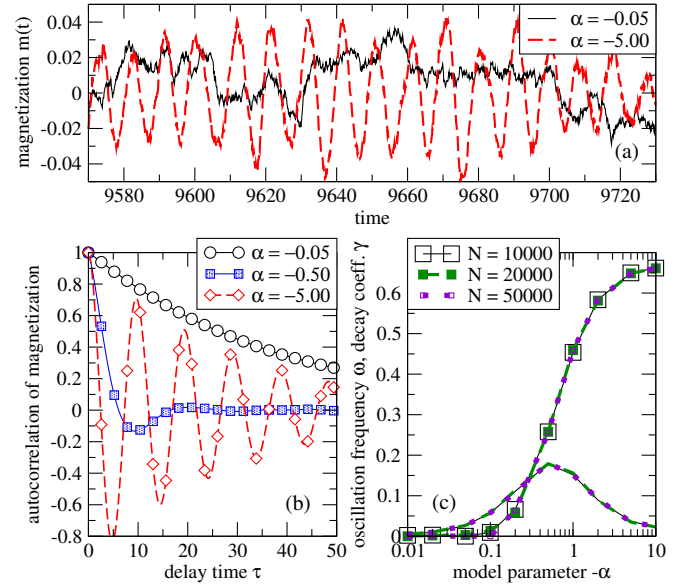


FIG. 4. Oscillations and decay of correlations for the finite-size model in the regime $\alpha < 0$. (a) Excerpts from time series of the magnetization for two different values of α , system size $N = 50000$. (b) Autocorrelation functions from magnetization time series of length $T = 10^5$, system size $N = 50000$. (c) Oscillation frequency ω (curves with symbols) and decay coefficients γ (no symbols) extracted from time series under different values of α and N . Curves for different system sizes N are almost indistinguishable. Autocorrelation functions $A(\tau)$, $\tau \in [0, 100]$, are considered for a least squares fit of the functional form $A_{\text{fit}}(t) = \exp(-\gamma t) \cos(\omega t)$.

$S^+, S^- \in [N]$ be the two sets of agents with equal opinion within each set and different opinions across sets. We define the *convincingness* of S^+ vs. S^- as the probability z that the interaction of a uniformly random pair of an S^+ agent and an S^- agent leads to adoption of the S^+ opinion,

$$z = |S^+|^{-1}|S^-|^{-1} \sum_{i \in S^+} \sum_{j \in S^-} \frac{\tau_i^\alpha}{\tau_i^\alpha + \tau_j^\alpha}. \quad (3)$$

In case $\alpha < 0$, there are competing effects governing the dynamics of z . When $i \in S^+$ convinces $j \in S^-$, i) the set S^+ gains another member who now has the youngest opinion increasing z . ii) The set S^- loses a member j with τ_j typically larger than average, making opinions of S^- members younger on average decreasing z . iii) With time advancing, all opinions age by the same additive rate. This makes ratios between ages smaller, driving z towards $1/2$. In the case $\alpha \ll -1$, the first effect dominates. Thus, an initial advantage in z is amplified and the system orders quickly. For $\alpha \approx -1$, ordering times are large due to dominance of the second and third effects. In order to verify this idea, we numerically record data pairs $(z(t), z(t+\tau) - z(t))$ with $\tau = 1$. Averaging over pairs with the same or similar $z(t)$ values, we obtain $\langle z(t+\tau) - z(t) \rangle$ as the expected restoring force. The corresponding standard deviation is the noise strength at this z value. The restoring force for z is linear around the equilibrium point $z(t) = 0.5$ while the noise strength is mostly independent of z (see Fig. 1 at Supplemental Material). This suggests to picture the dynamics around $\alpha = 0$ as one-dimensional equilibrium in a hyperbolic potential under state-independent additive noise.

Different real systems display dominance such as in the adoption of innovations [4, 5] and alternation as in opinion formation dynamics [41–43] or economic cycles [44]. As an example, Fig. 5 shows the electoral results of the governmental elections for United States, United Kingdom, and Canada during several decades [45]. The Lomb periodogram [46] of the binary time series reveals the existence of alternation between the political parties, by the presence of prominent peaks well above the noise level (shuffling of the data), with periods of 20–30 years in agreement with observations [47]. This period of time coincides approximately with the length of a generation. Different mechanisms have been proposed to explain political cycles: electorate disappointment [48], voters mood changes [47, 49], negativity effect [50] or policy inertia [51]. Our study complements those mechanisms and contributes to understand how a continuous age state dynamics model with competition between preference for old versus young opinions leads to alternation in the leading opinion.

Summarizing, we have studied the competition of states using a basic model that takes into account the aging of the state. The stability analysis of the solutions

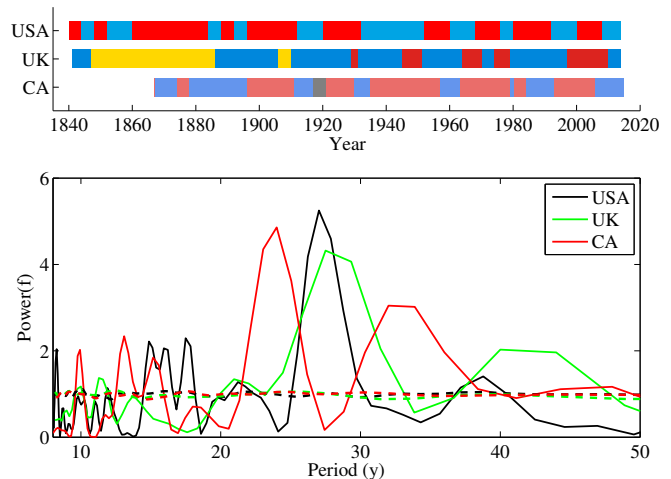


FIG. 5. Election results for United States, United Kingdom, and Canada. Elected parties are represented by their official colors, USA: Republicans (red) and Democrats (blue); UK: Conservative (blue), Liberals (yellow), and Labour (red); Canada: Liberals (red), Unionist (gray), and Conservatives (blue). Bottom panel: Lomb periodograms of the binary time series for each country. Dashed lines represents the level of noise as the result of shuffling the data 250 times.

reveals the existence of two stable solutions for positive values of the persuasiveness (old opinion prevails) that compete for consensus. For large negative persuasiveness (young opinion prevails), only one solution is stable leading to oscillatory transients. We have extended our study to a more detailed continuous age model finding that, when confronting two opinions, the final configuration where only one opinion survives and the time needed to reach it is noticeably sensitive to the age of the state through the exponent α of the convincing probability. The continuous age model exhibits likewise the oscillations shown by the basic model for large negative persuasiveness as well as similar solutions with α . Our study provides an alternative mechanism in the understanding of the dynamics of consensus formation and the observed alternation between states of different systems.

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